PHASE SEGREGATION DYNAMICS IN PARTICLE SYSTEMS WITH LONG RANGE INTERACTIONS II: INTERFACE MOTION.

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Abstract. We study properties of the solutions of a family of second order integro-differential equations, which describe the large scale dynamics of a class of microscopic phase segregation models with particle conserving dynamics. We first establish existence and uniqueness as well as some properties of the instantonic solutions. Then we concentrate on formal asymptotic (sharp interface) limits. We argue that the obtained interface evolution laws (a Stefan–like problem and the Mullins–Sekerka solidification model) coincide with the ones which can be obtained in the analogous limits from the Cahn–Hilliard equation, the fourth order PDE which is the standard macroscopic model for phase segregation with one conservation law.

1. Introduction

In part I [19] we rigorously derived a macroscopic equation describing the time evolution of the empirical average process, i.e. the local density, for a model of interacting particles with one conservation law undergoing phase segregation. Here we establish several properties of the solutions of a class of such equations, which are expected to describe more general microscopic models (see Section 3 of [19]). Our emphasis is on the late stages of phase segregation, when the phenomena are dominated by the motion of sharp interfaces separating domains of different densities.

Let us briefly recall in an informal way the results in part I. The particle models are dynamic versions of lattice gases with long range Kac potentials. By a long range Kac potential, we mean that the interaction energy between two particles, say between a particle at x and one at y (x and y are both in $\mathbb{Z}^d$), is given by $\gamma^d J(\gamma(x-y))$, where J is a smooth compactly supported function ($J(r) = J(-r)$) and γ is a positive parameter which is sent to zero. The equilibrium states for these models have been investigated thoroughly ([21],[26], [30]) and have provided great mathematical insight into the static aspects of phase transition phenomena. The dynamical version of these models, in which each particle jumps at random

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times from a site of the lattice $\mathbb{Z}^d$ to one of its unoccupied nearest neighbor sites is sometimes called *local mean field Kawasaki dynamics*. The jump times are chosen according to a probability distribution which depends on the particle configuration and is reversible with respect to the equilibrium Gibbs measure. To find a hydrodynamic scaling limit, we scale also the lattice spacing with γ and the time with γ^{-2} (this is the so called *diffusive limit*). We then derive a (deterministic) evolution law for the macroscopic density ρ.

In [19] we argue, but do not prove, that our results extend to the case in which there is a local (i.e. short range) interaction between the particles in addition to the Kac potential: the local interaction should however be sufficiently weak, more precisely the system with the local interaction *alone* should be in a high temperature regime. In the present paper we shall consider the equations expected in this more general case. They are given in terms of a second order integrodifferential equation:

$$\partial_t \rho(r,t) = \nabla \cdot \left[\sigma(\rho) \nabla \left(\frac{\delta \mathcal{F}}{\delta \rho} \right) \right] \tag{1.1}$$

ρ is the density, σ (a function of ρ) is the mobility and

$$\mathcal{F}(\rho) = \int_{T^d} f_c(\rho(r))\mathrm{d}r + \frac{1}{4} \int \int_{T^d \times T^d} J(r-r') \left[\rho(r) - \rho(r')\right]^2 \mathrm{d}r\mathrm{d}r' \tag{1.2}$$

in which f_c is either convex or it has a symmetric double well structure, with minima at ρ^- and ρ^+. The latter will be the relevant case for us: we will call the densities ρ^- and ρ^+ the *phases* of the system. The dependence on the temparature (of σ and f_c) is suppressed. In probabilistic terms, (1.1) is a law of large numbers for the empirical averages over the particle system. Equation (1.1) is in a particularly enlightening form: it is the gradient flux associated with the classical local mean field free energy functional (1.2) ([26,30]), with a density dependent mobility σ. The form of equation (1.1) allows us to connect the concepts of critical temperature, phases, stable, unstable and metastable states of the model, with the properties of the solutions of the evolution equation. The next step is to understand the evolution of the boundaries (interfaces) between regions in which the density is close to $\rho^{\pm}$.

Formal matched asymptotic expansions, in the so called *sharp interface limit*, of the solution of macroscopic evolution equations (see e.g. [24,29,5,6]) have been successfully employed to understand the interface motion in several models. By *sharp interface* we mean the limit in which the phase domains are very large with respect to the size of the interfacial region: if we denote by L the *typical* size of the domains, we will look for results in the limit $L \to \infty$. The time will have to be properly scaled as well, typically as some integer power of L, according to the type of initial condition (see Section 3).

The general picture that we obtain for the interface motion is the following: choose an initial condition which takes only metastable or stable values over large domains(of typical diameter $O(L)$), while the interfacial regions are layers of thickness $O(1)$ and let it evolve according to (1.1). There is, first, the equilibration of the interface which happens on a fast time scale ($t \ll L^2$). Then, on the time scale $t \propto L^2$ the evolution of the density in the bulks (that is the interior of the domains) is given by a nondegenerate nonlinear diffusion equation with Dirichlet boundary conditions on a free boundary, the interface (*Stefan problem*). Once the density

in the bulks is relaxed to the density of the phases, the motion of the interface on this time scale stops. A slower evolution can then be seen on the time scale $t \propto L^3$ and the motion of the interfaces is given by the Mullins–Sekerka model; a quasistatic free boundary problem in which the mean curvature of the interface plays a fundamental role.

Throughout the paper we will repeatedly stress the fact that the evolution in the sharp interface limits we have just outlined are the same as those obtained from the Cahn–Hilliard equation in the corresponding limits. The Cahn–Hilliard equation [7] is an equation of the form (1.1) but the free energy functional is chosen of the Ginzburg–Landau type

$$\mathcal{F}_{GL}(\rho) = \int \left[V(\rho) + |\nabla \rho|^2\right] \mathrm{d}r \tag{1.3}$$

in which V is a smooth function (say a polynomial) with a double minimum since the aim is to study phase segregation phenomena. The results on sharp interface motion for the Cahn–Hilliard equation have been obtained formally by R. L. Pego [29] and part of his program has been made rigorous recently [1,3]. In [29] only the case $\sigma = 1$ is considered, but the general case is analogous as long as the mobility does not vanish in the phases (the minima of V). The differences between the two models in this case are mostly in their having different surface tension, which appears as a factor in the formulas: the parameters of the models can therefore be tuned to give exactly the same sharp interface motion. A possibly deeper difference arises in the case of zero temperature (in our case the temperature is built in the model while in the Cahn–Hilliard case one has to choose an appropriate V which is temperature dependent, see e.g. [6]): the profile of the stationary front solutions (*instantonic solutions*) in the two cases has a substantially different limit. The instantonic solution in our case approaches a step function, while in the C–H case it approaches a differentiable function. The zero temperature case has been treated (formally) in [6], but here we will limit ourselves to a few observations in this case.

Finally we remark that in the case of particle dynamics without conservation law (Glauber or Kawasaki+Glauber dynamics [32]) there are by now several results proving that the interface motion is by mean curvature ([2,12,22,33]).

The paper is organized as follows: in Section 2 we introduce in an informal way the equations and all the relevant thermodynamical concepts. In Section 3 we introduce the sharp interface limit and state the results of the formal asymptotics: the computations are carried out in Section 5. In Section 4 we state and prove existence and uniqueness for the equations introduced in Section 2 as well as a result of existence and uniqueness of *istantonic* solutions.

2. The nonlocal equation

In this Section we introduce the evolution equations in an informal way, assuming that all the functions which appear in the formulas are suitably smooth. Moreover, in order to understand better the role of the quantities appearing in the definitions, we will restrict ourselves for the moment to the cases in which we will study phase segregation phenomena. Many of the assumptions made here are not needed to establish properties like existence and uniqueness: these will be stated and proven in a more general context in Section 4.

Let $T^d = \mathbb{R}^d \bmod a\mathbb{Z}^d$ be the torus of diameter $a \in \mathbb{R}^+$ (seen as a metric space, equipped with the periodic Euclidean distance). Occasionally we will consider the case $a = +\infty$, that is $T^d = \mathbb{R}^d$, but, unless esplicitly stated, a has to be considered fixed (and finite).

The free energy and the mobility. We start by introducing the free energy functional

$$\mathcal{F}(\rho) = \int_{T^d} f(\rho(r))\mathrm{d}r - \frac{1}{2}\int\int_{T^d \times T^d} J(r-r')\rho(r)\rho(r')\mathrm{d}r\mathrm{d}r' \tag{2.1}$$

in which ρ is a function from T^d to $[0,1]$, $f : [0,1] \to \mathbb{R}$ is convex and we will refer to $f(\rho)$ as the *free energy of the reference system* corresponding to the *interaction potential* J being zero. J is chosen *attractive*,

$$J \geq 0, \tag{2.2}$$

isotropic,

$$J(r) \text{ depends only on } |r| \tag{2.3}$$

and is compactly supported in a ball of diameter smaller than the diameter of T^d.

Define moreover the *mobility* σ, a function from $[0,1]$ taking nonnegative values and such that $\sigma(0) = \sigma(1) = 0$.

Further assumptions on f and σ. We will make three further assumptions on f and σ:
1. *Symmetry.* We assume that both f and σ are symmetric with respect to 1/2, i.e.

$$f((1+m)/2) = f((1-m)/2), \qquad \sigma((1+m)/2) = \sigma((1-m)/2) \tag{2.4}$$

for all $m \in [-1,1]$. This is a consequence of the particle–hole symmetry in the microscopic dynamics.
2. *The reference system is a nondegenerate diffusion.* We will require that there exists $c \in [1,\infty)$ such that

$$\frac{1}{c} \leq D(\rho) \equiv \sigma(\rho) f''(\rho) \leq c \tag{2.5}$$

for all $\rho \in (0,1)$. We will assume that the limits of $D(\rho)$ when ρ approaches 0 and 1 exist and these will extend the definition of $D(\cdot)$ to $[0,1]$.
3. *There are at most two phases.* If we set

$$f_c(\rho) = -\frac{\hat{J}(0)}{2}\left(\rho - \frac{1}{2}\right)^2 + f(\rho) \tag{2.6}$$

where $\hat{J}(0) = \int J(r)\mathrm{d}r$, the free energy (2.1) becomes, up to a irrelevant additive constant, the functional defined in (1.2). If ρ is constant the second term in (1.2) vanishes, so that f_c is the free energy density of a homogeneous profile (or *constrained equilibrium* free energy, see [26, 30]). For (small enough) values of $\hat{J}(0)$, f_c can be convex: in this case there is no phase segregation. We are then focusing on

the nonconvex case and we assume that f_c has a double well structure: precisely, there exists ρ_m^+, $\rho_m^- \in (0,1)$, $2\rho_m^+ - 1 = 1 - 2\rho_m^-$, such that

$$\begin{cases} f_c''(\rho) < 0, & \text{if } \rho \in (\rho_m^-, \rho_m^+) \\ f_c''(\rho) > 0, & \text{if } \rho \in (0, \rho_m^-) \cup (\rho_m^+, 1). \end{cases} \tag{2.7}$$

Once f_c is non convex, assumption (2.7) is, for example, consequence of the stronger assumption

$$f''(\rho) \text{ is strictly increasing for } \rho > 1/2 \tag{2.8}$$

which holds if the reference system is ferromagnetic (see Section 1 of [25]).

The evolution problem. We consider the equation

$$\partial_t \rho = \nabla \left[\sigma \nabla \left(\frac{\delta \mathcal{F}}{\delta \rho} \right) \right] \tag{2.9}$$

with initial condition $\rho(\cdot, 0) = \overline{\rho}(\cdot)$. An alternative way to write (2.9) is

$$\partial_t \rho(r,t) = \nabla \left[D(\rho(r,t)) \nabla \rho(r,t) - \sigma(\rho(r,t)) (\nabla (J(\cdot) * \rho(\cdot, t))(r) \right] \tag{2.10}$$

where $*$ denotes the convolution in T^d. Global existence and uniqueness for a weak version of (2.10) is given in Theorem 4.1; see Remark 4.1 for the regularity properties.

Stable, unstable and metastable states. By assumption (2.7) the interval $(0,1)$ can be partitioned into three sets:
(1) the **unstable region**

$$[\rho_m^-, \rho_m^+] \tag{2.11}$$

in which $f_c'' \le 0$ and $\rho_m^\pm$ are defined by $f_c''(\rho_m^\pm) = 0$;
(2) the **metastable region**

$$[\rho^-, \rho_m^-) \cup (\rho_m^+, \rho^+] \tag{2.12}$$

where $\rho^\pm$ are defined by $f_c'(\rho^\pm) = 0$;
(3) the **stable region**

$$[0, \rho^-) \cup (\rho^+, 1] \tag{2.13}$$

in which f_c'' is bounded away from zero (and it is $+\infty$ at 0 and 1). The two minima $\rho^\pm$ are called *phases*.

All the definitions and assuptions we have made are consequences of the statistical mechanical origin of the model and they are explained at length in [18] and [19]. In particular the fact that the density ρ is between 0 and 1 follows from the original particle model which cannot have more than one particle at a lattice site. Another consequence of this is that $\sigma(0) = \sigma(1) = 0$. The free energy f comes from a particle model which is above its critical temperature and it is then a smooth function in $(0,1)$ [31]. The control on the regularity of the mobility σ is harder, but Lipschitz continuity (see (4.8)) can be shown to hold in a rather general context [32]. The assumption (2.5) is natural, since D is the diffusion coefficient of a high temperature system (see [19] and [32]). We stress however that there is no fundamental reason to assume attractivity and isotropy of $J(r)$ to derive (2.9). Even in a non attractive case interesting phase segregation phenomena might occur: this is not considered in this paper. Moreover, in the anisotropic case one can do a formal asymptotic analysis as in Section 5 and get more general results than the ones we present; see (4.9) for a condition which replaces the isotropy.

A particular case. For the particle system treated in [19] with complete proofs we have free energy density

$$f(\rho) = \frac{1}{\beta}(\rho \log \rho + (1-\rho)\log(1-\rho)) \tag{2.14}$$

and mobility

$$\sigma^0(\rho) = \beta\rho(1-\rho) \tag{2.15}$$

where β is a positive parameter, corresponding to the inverse of the temperature. In this case, for $\beta\hat{J}(0) > 1/4$, $f_c''(1/2) < 0$, while for $\beta\hat{J}(0) < 1/4$ we have that $f_c''(\rho) > 0$ for all $\rho \in (0,1)$. Notice that in this example $D = 1$.

3. The sharp interface limit

We want to study the motion of boundaries between regions in which the density is either stable or metastable. Our analysis is based on formal matched asymptotic expansions.

Space–time rescaling. in what follows we shall consider the evolution (2.9) with r in $\epsilon^{-1}T^d$ with $\epsilon > 0$ and we will scale time as ϵ^{-q}, with $q = 0, 2, 3$ (only a short remark will be made about $q = 4$), according to the case. We will make the substitution $t = \epsilon^{-q}\tau_q$. At this stage, it is more convenient to reformulate the problem in terms of the rescaled densities

$$\rho^\epsilon(r, \tau_q) = \rho(\epsilon^{-1}r, \epsilon^{-q}\tau_q) \tag{3.1}$$

in which the spatial variable r ranges in T^d and τ_q is a positive number. The evolution equation (2.9) yields

$$\epsilon^{q-2}\partial_{\tau_q}\rho^\epsilon(r,\tau_q) = \nabla\left[\sigma(\rho^\epsilon(r,\tau_q))\nabla\mu^\epsilon(r,\tau_q)\right] \tag{3.2}$$

in which

$$\mu^\epsilon(\rho^\epsilon)(r,\tau) = f'(\rho^\epsilon(r,\tau)) - \int_{T^d} J_\epsilon(r-r')\rho^\epsilon(r',\tau)\mathrm{d}r' \tag{3.3}$$

and $J_\epsilon(r) = \epsilon^{-d}J(\epsilon^{-1}r)$. Recall that J is taken to be supported in a fixed ball, independent of ϵ.

Before starting the analysis of the time evolution of the system, we need to introduce the concept of instantonic solution.

The one dimensional problem and its instantonic solution. We replace T^d with $\mathbb{R}^d$ in (2.9) and let the initial condition $\rho^0(r) = u_0(r_1)$ depend only on the $1-$coordinate. Then (2.9) reduces to the one dimensional problem ($r_1 = z$)

$$\partial_t u(z,t) = \partial_z\left[\sigma(u(z,t))\left(\partial_z f'(u(z,t)) - \partial_z((\tilde{J}*u)(z,t)\right)\right] \tag{3.4}$$

in which

$$\tilde{J}(z) = \int_{\mathbb{R}^{d-1}} J(z, r_2, \ldots, r_d)\mathrm{d}r_2\ldots\mathrm{d}r_d \tag{3.5}$$

The initial condition is $u(\cdot,0) = u_0(\cdot)$. Clearly $u(z,t) \equiv C \in [0,1]$ is a stationary solution. It is easy to see that, if C is in the unstable region, this solution is unstable.

In Section 4, Proposition 4.2, we show that (3.4) admits a nonconstant stationary solution which is unique up to translation and reflections in a proper class (see Proposition 4.2 for the precise statement). U solves the functional equation (see the proof of Proposition 4.2)

$$U(z) = (f')^{-1}\left(\left(\tilde{J} * \left(U - \frac{1}{2}\right)(z)\right)\right) \tag{3.6}$$

in which $(f')^{-1}$ is the inverse function of f', which is invertible since we assumed $f'' > 0$. From (3.6) it follows directly that

$$\lim_{z\to\pm\infty} U(z) = \rho^{\pm} \tag{3.7}$$

hence the asymptotic values for U, which we will call the instantonic solution, are exactly the phases. We observe that $\rho(r,t) = U(z_0 + \nu \cdot r)$ is a stationary solution of (2.9) in the case $T^d = \mathbb{R}^d$ for any choice of $z_0 \in \mathbb{R}$ and $\nu \in \mathbb{R}^d$, $|\nu| = 1$. This solution represents a plane stationary front.

The sharp interface initial condition. The type of initial density profiles ρ_0^ϵ that we wish to consider take values which are either metastable or stable on the whole space T^d, with the exception of the points which are at distance $O(\epsilon)$ from a smooth (hyper)surface $\Gamma_0 \subset T^d$, which will be called the interface. We will denote by $\nu(r)$ one of the two unit vectors normal to the interface at the point r: the (conventional) choice of the orientation will be explained below. We suppose that there exists a function ρ_0, defined in T^d and smooth outside Γ_0, such that if r is not on the interface, then

$$\lim_{\epsilon\to 0} \rho_0^\epsilon(r) = \rho_0(r) \tag{3.8}$$

and the corresponding limit holds also for the derivatives of ρ_0^ϵ. Moreover if $\phi(r,\Gamma_0) = O(\epsilon)$ (ϕ is the signed distance), then

$$\rho_0^\epsilon(r) = U(\epsilon^{-1}\phi(r,\Gamma_0)) + O(\epsilon) \tag{3.9}$$

In particular the limit of $\rho_0(r)$ as r approaches Γ_0 from the interior of a cluster is either ρ^+ or ρ^- and ρ_0 is discontinuous at Γ_0. The sign of $\phi(r,\Gamma_0)$ is chosen to be positive (respectively negative) if $\rho_0(r) > 1/2$ (respectively $\rho_0(r) < 1/2$) and ν is chosen to point in the direction in which ϕ increases, i.e. $\nu = \nabla\phi$ on the interface.

It turns out that the relevant time scale for the evolution of Γ_0 is $t \propto \epsilon^{-2}$, that is $q = 2$ in (3.2).

A remark about the equilibration of the interface. The sharp interface initial condition that we introduced is such that the interface is locally stationary, since it is locally very close to the plane stationary front. It is however very natural to consider initial condition which satisfy (3.8), but not necessarily (3.9). In particular, the limiting valuess of ρ_0 on the interface may not coincide with the phases. In this case, the first observation is that $\partial_t\rho^\epsilon(r,t) = O(\epsilon^2)$ for $t = 0$ and r away from the interface. However if r is close to the interface then $\partial_t\rho^\epsilon(r,t) = O(1)$ at $t = 0$. We then expect the transition layer around the interface to evolve on the time scale $O(1)$ and eventually (asymptotically on the time scale $O(1)$) to approach a profile which, to leading order, is given by $U(\epsilon^{-1}\phi(r,\Gamma_0))$. This is a strong *ansatz* on the

behavior of the solutions of (2.9). Due to the conservation of mass, an immediate problem arises: the two asymptotic values of U, i.e. the two phases, will not in general match with the solution away from the interface. This happens whenever the values of ρ_0 near Γ_0 do not coincide with the phases. In this case we have to deal with a boundary layer problem. This early stage of the evolution can be studied, at least on a formal level, and it gives rise to a self similar Stefan problem: the formal expansion is absolutely identical to that in Section 3 of [29]). We will not focus on this stage of the evolution and we will only remark that the boundary layer should be of thickness $O(\epsilon\sqrt{t})$, so that it disappears if $t = O(\epsilon^{-2})$. We will focus on the later stage by assuming that this stabilization has taken place.

Sharp interface limit I: Stefan problem ($q = 2$). If we assume that the interface is stable (see [1] for some rigorous statements on stability of interfaces for the Cahn–Hilliard model) the structure of the initial condition is essentially preserved and hence $\rho^\epsilon(r, \tau_2)$ (together with its derivatives) converges to $\rho(r, \tau_2)$ away from a hypersurface Γ_{τ_2} while the boundary values of $\rho(r, \tau_2)$ at Γ coincide with the phases. Moreover we assume that ρ is smooth outside Γ_{τ_2} and that Γ_{τ_2} is itself smooth and $(d-1)$–dimensional. In general this will be true only for τ_2 in a finite interval and when a singularity appears one should introduce a weak version of the evolution law. We have the natural domain decomposition

$$T^d = \Gamma_{\tau_2} \cup \Omega^+_{\tau_2} \cup \Omega^-_{\tau_2} \tag{3.10}$$

and the sets appearing on the right–hand side of (3.10) are disjoint.

Statement 1. *In the setting specified above and at the level of formal asymptotics, $\rho^\epsilon(r, \tau_2)$ converges to $\rho(r, \tau_2)$, which solves the following nonlinear Stefan problem*

$$\begin{cases} \partial_{\tau_2}\rho = \nabla \cdot (\sigma(\rho)\nabla\mu_0(\rho)) & \text{for } r \in \Omega^\pm_{\tau_2} \\ \lim_{r' \to (r)^\pm} \rho(r', \tau_2) = \rho^\pm & \text{for } r \in \Gamma_{\tau_2} \\ \rho(r, 0) = \rho_0(r) & \text{for all } r \in T^d \end{cases} \tag{3.11}$$

in which $\mu_0(\rho) = f_c'(\rho)$ and by limit as r' goes to $(r)^\pm$ we mean the limit $\lambda \to 0^\pm$ with $r' = r + \lambda\nu$. The motion of the interface is generated by the velocity field $V_2(r)\nu(r)$, where $r \in \Gamma_{\tau_2}$ and

$$V_2(r) = \frac{\sigma^\pm}{(2\rho^+ - 1)} \left[\nu \cdot \nabla\mu_0\right]^+_- (r) \tag{3.12}$$

in which $[\nu\cdot\nabla\mu_0]^+_-(r) = \lim_{r'\to(r)^+} \nu(r)\cdot\nabla\mu_0(r') - \lim_{r'\to(r)^-} \nu(r)\cdot\nabla\mu_0(r')$, i.e. the jump in the normal gradient of μ_0 across the interface and $\sigma^\pm = \sigma(\rho^+) = \sigma(\rho^-)$.

Remark 1: in the case $\Gamma_0 = \emptyset$, that is the density takes only stable or metastable values and there is no interface, this result has been made rigorous, even starting directly from the particle system (see references in [19]).

Remark 2: The Stefan problem we just introduced is expected to stabilize as $\tau_2 \to \infty$ and it should lead to homogeneous density in the (surviving clusters). This fact is however very nontrivial from the mathematical viewpoint and the problem (3.11), (3.12) may become ill–posed for some finite τ_2. It is however easy to verify that if the density in the clusters is that of one of the homegeneous phases, then

$V_2 \equiv 0$ and so there is no macroscopic motion of the interfaces on the time scale $q = 2$, but there may be a motion on a longer time scale, i.e. if q is larger.

The sharp interface II: the Mullins–Sekerka symmetric solidification model ($q = 3$). Let us assume that the initial condition satisfies the extra condition $\rho_0^\epsilon(r) = \rho^\pm + O(\epsilon)$ for every r away from the interface and that, like for the case $q = 2$, the density profile at the interface is given by the rescaled instantonic solution in the direction perpendicular to the interface. It turns out that, for such an initial condition, the relevant time scale is $q = 3$. Under the assumption of the stability of the interface (see the discussion for the $q = 2$ case), we can describe the time evolution of the density in terms of the evolution of the interface, since the density in the cluster is just the equilibrium one. The problem is then reduced to find the geometrical motion of the interface Γ_{τ_3}. The analog of the natural domain decomposition (3.10) clearly holds also in this case. In the Appendix it is shown that for this model the surface tension (i.e. the free energy density associated with the instantonic solution U, see [4]) is

$$S = \frac{1}{2} \int_{\mathbb{R}\times\mathbb{R}} (z' - z)\partial_z U(z) U(z') \tilde{J}(z - z') \mathrm{d}z \mathrm{d}z' \tag{3.13}$$

Let us furthermore denote by $K(r)$, r on the interface, the sum of the principal curvatures of the interface at r, that is $d - 1$ times the mean curvature at r.

Statement 2. *In the setting explained above and at the level of formal asymptotics, the motion of the interface Γ_{τ_3} is generated by the velocity field $V_3(r)\nu(r)$, for r belonging to the interface Γ_{τ_3}. The speed is given by the following expression*

$$V_3(r) = \frac{\sigma^\pm}{2\rho^+ - 1} [\nu \cdot \nabla \mu_1]_-^+(r) \tag{3.14}$$

where $[\nu \cdot \nabla \mu_1]_-^+(r)$ denotes the jump of the component normal to the interface of the function μ_1, which is continuous on T^d, at least twice differentiable on $\Omega_{\tau_3}^\pm$ and solves the static problem

$$\begin{cases} \Delta \mu_1(r) = 0 & \text{for } r \in \Omega_{\tau_3}^+ \cup \Omega_{\tau_3}^- \\ \mu_1(r) = -SK(r)/(2\rho^+ - 1) & \text{for } r \in \Gamma_{\tau_3} \end{cases} \tag{3.15}$$

Remark 1: The global existence for the Mullins–Sekerka evolution law presents severe mathematical difficulties. We refer to [8] for these aspects of the problem.

Remark 2: The function μ_1, which appears in (3.14) and (3.15), may be thought of just as an auxiliary tool for computing the interface velocity. It does however have a direct physical meaning: it is the $O(\epsilon)$ correction to the chemical potential away from the interface, i.e. $\mu = \mu(\rho^\pm) + \epsilon\mu_1 + o(\epsilon)$ in the interior of the clusters.

Some remarks on the limit of vanishing temperature. Here we restrict ourselves to the model defined by (2.14) and (2.15). To stress the dependence on the parameter β we add a subscript β to U, $\sigma^\pm$, $\rho^\pm$ and S. In this case, a natural question is what happens to the velocity V_3 in the Mullins–Sekerka model as β goes to infinity. In this limit the instantonic solution U_β approaches the Heaviside function (easily shown using formula (3.6)), i.e. the characteristic function of the positive semiaxis. Therefore, using (3.13), we easily see that

$$\lim_{\beta\to\infty} S_\beta = S_\infty \equiv \frac{1}{2} \int_0^\infty r\tilde{J}(r) dr$$

and S_∞ is strictly positive and finite. By (3.15) the gradient of μ_1 is uniformly bounded as long as the interface is smooth. So what really matters to our analysis is the behavior, as β becomes larger and larger, of the factor $\beta\sigma^\pm_\beta/(2\rho^+_\beta - 1)$ in (3.14). While the denominator tends to one, the mobility in the phase σ_β vanishes exponentially fast. Our conclusion is then

$$\lim_{\beta\to\infty} V_3 = 0$$

and the limit is approached exponentially fast. So the Mullins–Sekerka motion is rapidly frozen when the temperature is lowered. A motion on the time scale $q = 4$ is expected to arise (surface diffusion), but this will not be considered here.

4. Existence, uniqueness and regularity

We follow closely the notation of [10], to which we refer for all the basic notions and results. In the first part of this Section we work in T^d (unit torus), so that $L^2 = L^2(T^d)$, $H^1 = H^1(T^d), \dots$. $\tau > 0$ is fixed and our basic (Hilbert) space is

$$W \equiv W(0,\tau;\, H^1, H^{-1}) \equiv \left\{ w : w \in L^2(0,\tau; H^1),\ \frac{\mathrm{d}w}{\mathrm{d}t} \in L^2(0,\tau; H^{-1}) \right\} \tag{4.1}$$

where $H^{-1} = (H^1)'$, the dual of H^1 with respect to the L^2 scalar product $(\cdot,\cdot)$. The norm in H^1 will be denoted by $\|\cdot\|$, the norm in L^2 by $|\cdot|_2$ and the norm in H^{-1} by $\|\cdot\|_*$. In particular we have

$$\|v\|_* = \sup_{u\in H^1} \frac{(u,v)}{\|u\|} = |(1-\Delta)^{-1/2} v)|_2 = \left(\sum_{k\in 2\pi\mathbb{Z}^d} \frac{|\hat{v}(k)|^2}{1+k^2} \right)^{1/2} \tag{4.2}$$

in which $v \in H^{-1}$ and $\hat{v}$ is the Fourier transform of v. The space W is equipped with the (Hilbert) norm

$$\begin{aligned} \|w\|_W &= \left(\|w\|^2_{L^2(0,\tau;H^1)} + \|w'\|^2_{L^2(0,\tau;H^{-1})} \right)^{1/2} \\ &= \left(\int_0^\tau \left[\|w(t)\|^2 + \|w'(t)\|^2_* \right] \mathrm{d}t \right)^{1/2} \end{aligned} \tag{4.3}$$

in which $w' = \mathrm{d}w/\mathrm{d}t$. We will often work with the following convex subset of W

$$W_1 \equiv \{ w \in W : 0 \le w(t,x) \le 1 \text{ for almost all } (t,x) \in [0,\tau]\times T^d \}. \tag{4.4}$$

We say that $\rho \in W_1$ is a weak solution of (2.9) if for all $u \in H^1$

$$\frac{\mathrm{d}}{\mathrm{d}t}(u,\rho) + (\nabla u, D(\rho)\nabla\rho - \sigma(\rho)(\nabla J * \rho)) = 0 \tag{4.5}$$

in the sense of $\mathcal{D}'((0,\tau))$ ($\mathcal{D}((0,\tau))$ is the space of C^∞ compactly supported real functions over $(0,\tau)$) and

$$\rho(0) = \overline{\rho}. \tag{4.6}$$

We recall that (4.6) makes sense because W can be continuously imbedded in $C^0(0,\tau;L^2)$, Th.1, Ch.XVIII of [10], and that $\mathrm{d}\,(u,\rho)\,/\mathrm{d}t = (u,\rho')$.

The theorem we are going to state below holds under the assumptions made above and under the following assumptions on D, σ and J.

$$D \in C^0([0,1]) \tag{4.7}$$

($u \in C^{k-}$, $k \in \mathbb{Z}^+$, means that the derivatives of u of order $(k-1)^{th}$ are Lipschitz) and that $1/c \le D \le c$ for some $c \ge 1$. Moreover

$$\sigma \in C^{1-}([0,1]) \qquad \sigma(\rho) \ge 0 \;\text{ for all } \rho \in [0,1], \qquad \sigma(0) = \sigma(1) = 0. \tag{4.8}$$

Finally

$$J \in C^2(T^d), \qquad J(r) = J(-r) \text{ for all } r \in T^d. \tag{4.9}$$

Theorem 4.1. *There exists a unique solution to the problem* (4.5), (4.6)*.*

Proof.

Existence. We will make use of the following result.

Proposition 4.1. *For every $g \in L^2(0,\tau;L^2)$ the problem*

$$\begin{cases} u' = \nabla\{D(u)\nabla u - \sigma(u)(\nabla J * g)\} \\ \\ u(0) = \overline{\rho} \end{cases} \tag{4.10}$$

has a unique weak solution (in the same sense as (4.5) *and* (4.6)*).*

Proof of Proposition 4.1: (4.10) is a parabolic problem for which existence in W can be established via fixed point theorems, see e.g. [27], IV.4, (extending the definitions of σ and D outside $[0,1]$ in a arbitrary (nice) way). The comparison principle and the fact that $\sigma(0) = \sigma(1) = 0$, so that $u \equiv 1$ and $u \equiv 0$ are solutions of (4.10) with proper initial condition, immediately allows to obtain a solution in W_1. Uniqueness is also standard and can be, for example, established along the same line that we are going to use below to prove uniqueness for the original problem (4.5),(4.6). □

Proposition 4.1 defines a map $X : L^2(0,\tau;L^2) \to L^2(0,\tau;L^2)$

$$X(g) = u \tag{4.11}$$

and we notice that, since $0 \le u \le 1$ almost everywhere, $\|X(g)\|^2_{L^2(0,\tau;L^2)} \le \tau$. Directly from (4.10) one has that

$$(u(t), u'(t)) + \int_{T^d} |\nabla u(t)|^2 D(u(t)) = \int_{T^d} \sigma(u(t))(\nabla J * g(t))\nabla u(t)$$

so that by using the Cauchy–Schwarz and Young's inequalities, (2.5) and the fact that σ is bounded, one obtains that there is a finite constant c_1 such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|_2^2 + \frac{1}{2c}|\nabla u(t)|_2^2 \le \frac{c_1}{2}|g(t)|_2^2 \tag{4.12}$$

in which c is the constant in (2.5). From (4.12), by integrating in time, we extract the following bound

$$\int_0^\tau \|u(t)\|^2 \mathrm{d}t \leq c_2 \left[\|g\|^2_{L^2(0,\tau;L^2)} + 1\right] \tag{4.13}$$

for some finite constant c_2 (independent of the initial condition). Moreover

$$\|u'\|_* = \sup_{u_1 \in H^1 : \|u_1\|=1} \left(\int_{T^d} [\sigma(u)(\nabla J * g)\nabla u_1 - D(u)\nabla u \nabla u_1] \right)$$

which by the Cauchy–Schwarz inequality implies that there is a c_3 such that

$$\int_0^\tau \|u'(t)\|^2_* \mathrm{d}t \leq c_3 \left[\|g\|^2_{L^2(0,\tau;L^2)} + 1\right] \tag{4.14}$$

so that by (4.13) and (4.14) we have that there exists c_4 such that

$$\|u\|_W \leq c_4 \left[\|g\|^2_{L^2(0,\tau;L^2)} + 1\right] \tag{4.15}$$

which implies that X is compact, since W is compactly imbedded in $L^2(0,\tau;L^2)$, see Proposition 4.2, Section IV of [27]. X is also continuous since given a sequence $\{g_n\}$ which converges in $L^2(0,\tau;L^2)$ to g, the sequence $\{X(g_n)\} = \{u_n\}$ is relatively compact in $L^2(0,\tau;L^2)$ and weakly relatively compact in W (by (4.15)). This in particular implies that $D(u_n)\nabla u_n$ converges weakly along subsequences in $L^2(0,\tau;L^2)$ to $D(u)\nabla u$, since $D(\cdot)$ is C^0 and bounded. Hence any limit point u of $\{u_n\}$ (weakly in W and strongly in $L^2(0,\tau;L^2)$) solves (4.10), which has a unique solution, and the continuity of X follows.

Summing up, we have obtained that the continuous map X takes the closed convex set

$$\{u \in L^2(0,\tau;L^2) : \|u\|^2_{L^2(0,\tau;L^2)} \leq \tau\}$$

into itself and the image of this set under X is relatively compact (in particular $X(u) \in W_1$ and $\|u\|_W \leq c_4[1+\tau^{1/2}]$). By the Schauder fixed point Theorem ([20], Corollary 11.2) X has a fixed point $\rho \in W_1$ which satisfies (4.5), (4.6). This completes the proof of the existence.

Uniqueness.

Call ρ_1 and ρ_2 two solutions of (4.5) such that $\rho_1(0) = \rho_2(0)$. Set $\lambda(\rho) = \int_0^\rho D(\rho')\mathrm{d}\rho'$, so that $\lambda \in C^1$ and $\lambda'(\rho) \geq 1/c > 0$. We have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_1(t) - \rho_2(t)\|^2_* = -\left(\nabla(\rho_1(t) - \rho_2(t)), (1-\Delta)^{-1}\nabla(\lambda(\rho_1(t)) - \lambda(\rho_2(t)))\right) + \left(\nabla(\rho_1(t) - \rho_2(t)), (1-\Delta)^{-1}\nabla(\sigma(\rho_1)\nabla J * \rho_1(t) - \sigma(\rho_2)\nabla J * \rho_2(t))\right).$$

We then have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_1(t) - \rho_2(t)\|^2_* = -\left(\rho_1(t) - \rho_2(t), (\lambda(\rho_1(t)) - \lambda(\rho_2(t)))\right) + \left(\rho_1(t) - \rho_2(t), (1-\Delta)^{-1}(\lambda(\rho_1(t)) - \lambda(\rho_2(t)))\right)$$

$$+\left(\nabla(\rho_1(t)-\rho_2(t)),(1-\Delta)^{-1}\nabla(\sigma(\rho_1)\nabla J*\rho_1(t)-\sigma(\rho_2)\nabla J*\rho_2(t))\right).$$

Since $\lambda' \geq 1/c > 0$ and by the Young and Cauchy–Schwarz inequalities, we obtain that for any $\epsilon_1, \epsilon_2 > 0$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_1(t)-\rho_2(t)\|_*^2 \leq -\frac{1}{c}\,|\rho_1(t)-\rho_2(t)|_2^2+\frac{\epsilon_1^2}{2}\|\lambda(\rho_1)-\lambda(\rho_2)\|_*^2+\frac{1}{2\epsilon_1^2}\|\rho_1-\rho_2\|_*^2$$

$$+\frac{1}{2\epsilon_2^2}\|\nabla(\rho_1-\rho_2)\|_*^2+\frac{\epsilon_2^2}{2}\|\nabla[\sigma(\rho_1)\nabla J*\rho_1(t)-\sigma(\rho_2)\nabla J*\rho_2(t)]\|_*. \tag{4.16}$$

Since for all $v \in L^2$, $\|v\|_* \leq |v|_2$, for all $u \in H^1$, $\|\nabla u\|_* \leq |u|_2$, and

$$\|\nabla[\sigma(\rho_1)\nabla J*\rho_1(t)-\sigma(\rho_2)\nabla J*\rho_2(t)]\|_* \leq |[\sigma(\rho_1)\nabla J*\rho_1(t)-\sigma(\rho_2)\nabla J*\rho_2(t)|_2$$

$$\leq \left|[\sigma(\rho_1)-\sigma(\rho_2)]|\nabla J*\rho_1|\right|_2+\left|\sigma(\rho_2)|\nabla J*(\rho_2-\rho_1)|\right|_2 \leq c_5|\rho_1-\rho_2|_2$$

where c_5 depends only on the Lipschitz constant of σ and on $|\nabla J|_2$. From (4.16), choosing ϵ_1 and ϵ_2 small, we obtain that there is c_6 such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_1(t)-\rho_2(t)\|_*^2 \leq c_6\,\|\rho_1(t)-\rho_2(t)\|_*^2$$

which, since $\|\rho_1(0)-\rho_2(0)\|_*=0$, implies that $\|\rho_1(t)-\rho_2(t)\|_*=0$ for all $t\in[0,\tau]$ and, since $\rho_1,\rho_2\in W$, $\|\rho_1(t)-\rho_2(t)\|=0$ which implies the result. $\square$

Remark 4.1: We observe that since $\rho \in C^0(0,\tau;L^2)$, $\nabla J*\rho \in C^0(0,\tau;C^1)$ (since $J\in C^2$). Once we have a solution of (4.5), (4.6), we have also a solution of (4.10) for $g = J*\rho$. It is then a standard result (see e.g. [23]) that, if $\overline{\rho}\in C^2$ and if D and σ are C^1, (4.10) has a classical solution, which is unique in the class of functions considered in Theorem 4.1 and Proposition 4.1. Therefore under the same assuptions one has a classical solution for our original problem (4.5), (4.6). One can obtain further regularity properties by making further assumptions on D, σ and J.

We now establish the existence and uniqueness (in a proper sense) of the istantonic solution and we give some of its properties. Sometimes in what follows we keep the notation with partial derivatives also for functions of one variable, for uniformity.

Consider the one dimensional setting with $a=\infty$, i.e. we are working on $\mathbb{R}$. Assume (2.2), (2.4), (2.8) , that f_c is non convex and all the assumptions made for Theorem 1. For consistency with the sharp asymptotic computations, below we use $\tilde{J}$ for J in the one dimensional setting (recall (3.5)). Take $\tilde{J}$ to be compactly supported.

Proposition 4.2. *Under the assumptions stated above, there exists a function $U\in C^1(\mathbb{R})$ such that for all $v\in C_0^\infty(\mathbb{R})$*

$$\int_{\mathbb{R}}\partial_z v(z)\left[D(U)\partial_z U(z)-\sigma(U(z))\partial_z(\tilde{J}*U)(z)\right]dz=0. \tag{4.17}$$

U satisfies:
(1) $U-1/2$ is odd;

(2) $U'(z) > 0$ for all $z \in \mathbb{R}$;
(3) There exists $m > 0$ such that

$$\lim_{z \to \pm\infty} e^{m|z|} \left| U(z) - \rho^{\pm} \right| = 0. \tag{4.18}$$

Moreover if a non constant non decreasing function $u \in H^1_{loc}(\mathbb{R})$, $u(z) \in (0,1)$ for all $z \in \mathbb{R}$, satisfies (4.17) with $U = u$ for all $v \in C_0^\infty(\mathbb{R})$, then there exists $\overline{z} \in \mathbb{R}$ such that $u(z) = U(z - \overline{z})$.

Proof: take u as in the statement of the Proposition above. Then there exists $C \in \mathbb{R}$ such that

$$D(u)\partial_z u(z) - \sigma(u(z))(\tilde{J}' * u)(z) = C \tag{4.19}$$

for almost all z. Since the second term in the left–hand side vanishes as $z \to \infty$ and $D > 1/c$, then $C \geq 0$ and $\partial_z u(z) \geq cC/2$ for z sufficiently large, so that $C = 0$. Since $u(z) \in (0,1)$ for all z, $|f''|$ (recall (2.5)) is bounded on every compact subset of $\mathbb{R}$ and $\sigma(u) > 0$. Hence we have

$$\partial_z \left[f'(u(z)) - (\tilde{J} * u)(z) \right] = 0 \tag{4.20}$$

which, using the fact that $f'' > 0$, is equivalent to

$$u(z) = (f')^{-1} \left((\tilde{J} * (u - 1/2))(z) + h \right) \tag{4.21}$$

for some $h \in \mathbb{R}$. In [16] and [14] it is shown that, in the class of functions we are considering and for the particular case $f'(\rho) = \log(\rho/1-\rho)$, so that $(f')^{-1}(x) = (1/2)[\tanh(x/2) + 1]$ there is a solution to (4.21) in the case $h = 0$ and [14] it is unique in a class larger than the one we consider. Their approach can be repeated in our context, but the general result in [9], in particular Remark 5.2 (*ii*), keeping in mind that we have the symmetry condition (2.7), covers the case considered here, establishing thus existence and uniqueness for $h = 0$: property (1) follows from (2.7) as well. Property (2) is established by the argument on page 706 of [14] and property (3) is established by repeating the proof of Proposition 2.2 in [13] for the general f considered here. We have thus established the existence of the function U claimed in the Proposition and that U solves (3.6). We remark that in [11,14,13], $\hat{J}(0) = 1$ and J is multiplied by $\beta > 0$, while in our case the role of β is played by $\hat{J}(0)$, and the density profile is given in terms of the magnetization variable $m = 2\rho - 1$.

We are left with showing that there is no solution to (4.21) for $h \neq 0$. In [9] (see also [11]) it is proven that the evolution equation

$$\partial_t u(r,t) = -u(r,t) + (f')^{-1} \left((\tilde{J} * (u(\cdot,t)(r) - 1/2) + h \right) \tag{4.22}$$

has a unique solution in the class of travelling wave solutions if $f'(\rho) - \hat{J}(0)(\rho - 1/2) - h = 0$ has three distinct solutions and this solution has non zero speed for $h \neq 0$. In particular this implies that there is no solution with zero speed, i.e. stationary, in the case of three distinct roots. Due to (2.8), $f'(\rho) - \hat{J}(0)(\rho - 1/2) - h$ can have at most three roots. If it has only one root ρ_1, then u must be constant,

since the asymptotic values of u must be equal to ρ_1. We are then left with the case in which there is a root ρ_1 and a double root ρ_2. With no loss of generality let us assume that $\rho_2 > \rho_1$ so that $\lim_{z\to\infty} u(z) = \rho_2$. Due to (2.8) we have that

$$f'(u) - \hat{J}(0)(u - 1/2) - h = \tilde{J} * (u - 1/2) - \hat{J}(0)(u - 1/2) \geq 0 \tag{4.23}$$

whenever $u(z) \in [\overline{u}, \rho_2]$, for some $\overline{u} < \rho_2$, and the inequality is strict for $u(z) \in [\overline{u}, \rho_2)$. If $z_0 = \inf\{z : u(z) < \rho_2\} < \infty$, then a contraddiction is easily established by (4.23) with $z = z_0$. If $z_0 = \infty$, since u is monotonic, (4.23) is valid for all z sufficiently large. However this implies that u is unbounded, which establishes the desired contraddiction and this completes the proof of uniqueness. We remark that uniqueness holds with no need of (2.8) if we restrict ourselves to establish uniqueness in the class of functions which have asymptotic at $z = \pm\infty$ which are symmetric with respect to 1/2, since this imposes $h = 0$ in (4.21). $\square$

We just remark that the regularity of the istantonic solution can be easily read out from (4.21) in terms of the regularity of J and f.

5. Formal asymptotic expansions

The computations regarding Statement 1 and Statement 2 follow closely the scheme of R. L. Pego's work [29]. We start by expanding the density profile in the bulk (*outer expansion*) and then we will do the same close to the interface (*inner expansion*): we will finally match the values at the boundaries.

We recall that (2.2), (2.3) and (2.7) are assumed. With abuse of notation, $J(R)$ ($R \in \mathbb{R}^+$) will mean $J(r)$, for $|r| = R$. Until further notice, $\tau = \tau_q$, any value of q (see (3.1) and (3.2)).

The outer expansion: we write

$$\rho^\epsilon(r,\tau) = \rho_0(r,\tau) + \epsilon\rho_1(r,\tau) + \epsilon^2\rho_2(r,\tau) + \ldots \tag{5.1}$$

$$\sigma^\epsilon(r,\tau) \equiv \sigma(\rho^\epsilon(r,\tau)) = \sigma_0(r,\tau) + \epsilon\sigma_1(r,\tau) + \epsilon^2\sigma_2(r,\tau) + \ldots \tag{5.2}$$

$$\mu^\epsilon(r,\tau) = \mu_0(r,\tau) + \epsilon\mu_1(r,\tau) + \epsilon^2\mu_2(r,\tau) + \ldots \tag{5.3}$$

By Taylor expansion, using (3.3), one obtains directly

$$\sigma_0(r,\tau) = \sigma(\rho_0(r,\tau)) \tag{5.4}$$

$$\sigma_1(r,\tau) = \rho_1\sigma'(\rho_0) \tag{5.5}$$

$$\sigma_2 = \rho_2\sigma'(\rho_0) + \rho_1^2\frac{\sigma''(\rho_0)}{2} \tag{5.6}$$

$$\mu_0 = f'(\rho_0) - \hat{J}(0)\rho_0 \tag{5.7}$$

$$\mu_1 = f''(\rho_0)\rho_1 - \hat{J}(0)\rho_1 \tag{5.8}$$

$$\mu_2 = f'''(\rho_0)\rho_2 + f''\frac{\rho_1^2}{2} - \hat{J}(0)\rho_2 - \frac{\bar{J}}{2}\Delta\rho_0 \tag{5.9}$$

in which $\bar{J} = (1/d)\int J(r)r^2\mathrm{d}r$ and we recall that $\hat{J}(0) = \int J(r)\mathrm{d}r$

The inner expansion: We want to expand the density profile ρ^ϵ in a small neighborhood of the interface Γ_τ, $\tau \geq 0$. One possible choice of the neighborhood is

$$\mathcal{N}_\epsilon = \{r \in \mathbb{R}^d : |\phi(r, \Gamma_\tau)| \leq \epsilon^a\} \tag{5.10}$$

where $a \in (0,1)$ (recall that ϕ denotes the signed distance). let us set $z = \phi(r, \Gamma_\tau)/\epsilon$. If ϵ is sufficiently small (depending on the minimum radius of curvature of the interface) then for each $r \in \mathcal{N}_\epsilon$ there is a unique $x = x(r) \in \Gamma_\tau$ such that $r = x + \nu(x)\epsilon z$. Notice that

$$\nu(x(r)) = \nabla\phi(r, \Gamma_\tau) \tag{5.11}$$

for any $r \in \mathcal{N}_\epsilon$. Moreover we define

$$V(x) \equiv V(x(r)) = \frac{\partial \phi}{\partial \tau}(r, \tau) \tag{5.12}$$

Note that $\partial\phi/\partial\tau$ is independent of $r \in \mathcal{N}_\epsilon$, as long as $x(r)$ is held fixed. For $r \in \mathcal{N}_\epsilon$ will look for expansions of the form

$$\rho^\epsilon(r, \tau) = \tilde{\rho}_0(z, r, \tau) + \epsilon\tilde{\rho}_1(z, r, \tau) + \epsilon^2\tilde{\rho}_2(z, r, \tau) + \ldots \tag{5.13}$$

$$\sigma^\epsilon(r, \tau) = \tilde{\sigma}_0(z, r, \tau) + \epsilon\tilde{\sigma}_1(z, r, \tau) + \epsilon^2\tilde{\sigma}_2(z, r, \tau) + \ldots \tag{5.14}$$

$$\mu^\epsilon(r, \tau) = \tilde{\mu}_0(z, r, \tau) + \epsilon\tilde{\mu}_1(z, r, \tau) + \epsilon^2\tilde{\mu}_2(z, r, \tau) + \ldots \tag{5.15}$$

in which any quantity $w(z, r, \tau)$ is defined for every r in $\mathcal{N}_\epsilon$ and $w(z, r + c\nu, \tau) = w(z, r, \tau)$, $\nu = \nu(x(r))$, for all $c \in \mathbb{R}$ such that $r + c\nu \in \mathcal{N}_\epsilon$, so that $\nu \cdot \nabla w(z, r, \tau) = 0$. The following formulas will be useful

$$\nabla\rho(r, \tau) = \epsilon^{-1}\nu(r, \tau)\partial_z\tilde{\rho}(z, r, \tau) + \nabla_r\tilde{\rho}(z, r, \tau) \tag{5.16}$$

$$\partial_\tau\rho(r, \tau) = \partial_\tau\tilde{\rho}(z, r, \tau) + \epsilon^{-1}V(r, \tau)\partial_z\tilde{\rho}(z, r, \tau) \tag{5.17}$$

The inner expansion for σ is identical to the outer one

$$\tilde{\sigma}_0 = \sigma(\tilde{\rho}_0) \tag{5.18}$$

$$\tilde{\sigma}_1 = \sigma'(\tilde{\rho}_0)\tilde{\rho}_1 \tag{5.19}$$

$$\tilde{\sigma}_2 = \sigma''(\tilde{\rho}_0)\tilde{\rho}_2 + \sigma'(\tilde{\rho}_0)\frac{\tilde{\rho}_1^2}{2} \tag{5.20}$$

More complicated is the computation of $\tilde{\mu}_i$: for the moment we give it only to first order

$$\tilde{\mu}_0 = f'(\tilde{\rho}_0) - \tilde{J} \star \tilde{\rho}_0 \tag{5.21}$$

in which $\star$ stands for the convolution in the z variable, that is

$$(\tilde{J} \star w)(z, r, \tau) = \int_{\mathbb{R}} \tilde{J}(z - z')w(z', x, \tau)\mathrm{d}z'$$

and we recall that $\tilde{J}$ is defined in (3.5).

The relation between Γ_τ and $\tilde\rho_i$ is given by the following *front centering condition* [29]:

$$\int_{-\infty}^{+\infty} U'(z)\,[\tilde\rho(z,r,\tau)-U(z)]\,\mathrm{d}z = 0 \tag{5.22}$$

where $\tilde\rho = \sum_i \epsilon^i \tilde\rho_i$ and the coefficient of each power of ϵ in (5.22) is required to vanish.

Matching conditions: We will match the outer and the inner expansion of the chemical potential order by order by requiring formally that

$$(\mu_0+\epsilon\mu_1+\epsilon^2\mu_2+\dots)(r+\epsilon z\nu(r,\tau),\tau) \approx (\tilde\mu_0+\epsilon\tilde\mu_1+\epsilon^2\tilde\mu_2+\dots)(z,r,t) \tag{5.23}$$

with $z=\epsilon^{1-a}$, $a\in(0,1)$. By expanding the left–hand side of (5.23) we obtain (see [29])

$$\mu_0^\pm(r,\tau) = \lim_{z\to\pm\infty} \tilde\mu_0(z,r,\tau) \tag{5.24}$$

$$(\mu_1^\pm + z\nu\cdot\nabla\mu_0^\pm)(r,\tau) = \tilde\mu_1(z,r,\tau)+o(1) \quad \text{as } z\to\pm\infty \tag{5.25}$$

$$\left(\mu_2^\pm + z\nu\cdot\nabla\mu_1^\pm + \frac{1}{2}z^2(\nu\cdot\nabla)^2\mu_0^\pm\right)(r,\tau) = \tilde\mu_2(z,x,\tau)+o(1) \quad \text{as } z\to\pm\infty \tag{5.26}$$

in which $\mu_i^\pm(r,\tau)=\lim_{t\to 0^\pm}\mu_i(r+\nu t,\tau)$, and analogous formula for the derivatives.

Verification of Statement 1: the case $q=2$. The starting point is equation (3.2) and we set $\tau=\tau_2$. By using (5.11), (5.16) and (5.17), we obtain that the highest order arising in the inner expansion of (3.2) ($O(\epsilon^{-2})$) yields the equation

$$\partial_z\left(\tilde\sigma_0\partial_z\tilde\mu_0\right)=0. \tag{5.27}$$

Using now the explicit expressions for $\tilde\sigma_0$ (5.18) and $\tilde\mu_0$ (5.21) we see that the solution $\tilde\rho_0$ of (5.27) coincides with the stationary solution of (3.4), that is

$$\tilde\rho_0(z,r,\tau)=U(z) \tag{5.28}$$

Moreover by using (3.6) we obtain

$$\tilde\mu_0(z,r,\tau)=\frac{\hat J(0)}{2} \tag{5.29}$$

The leading order of (3.2) in the outer expansion ($O(1)$) is

$$\partial_\tau\rho_0 = \nabla\left(\sigma_0\nabla\mu_0\right) \tag{5.30}$$

which by the assumptions on (5.4),(5.7) (see (2.7)) and on the initial condition is a nondegenerate parabolic PDE. (5.30) has to be supplemented with the boundary conditions coming from the matching rule (5.24) and the asymptotic value of $\tilde\rho_0$, given by (3.7). This completes the derivation of (3.11) and we are left with the identification of the speed of the interface. To do this we have to look at the next

order in the inner expansion ($O(\epsilon^{-1})$). From (5.12), (5.16), (5.17) and the fact that $\tilde{\mu}_0$ is constant (5.29) we obtain

$$V\partial_z\tilde{\rho}_0 = \partial_z\left(\tilde{\sigma}_0\partial_0\tilde{\mu}_1\right) \tag{5.31}$$

Recall that $\tilde{\rho}_0 = U$ and integrate in $z \in \mathbb{R}$ to obtain

$$V_2 \equiv V = \frac{\sigma^{\pm}\left[\partial_z\tilde{\mu}_1\right]_{-\infty}^{+\infty}}{[U]_{-\infty}^{+\infty}} \tag{5.32}$$

in which $[w]_{-\infty}^{+\infty} = w(+\infty, r, \tau) - w(-\infty, r, \tau)$ and recall that $\sigma(\rho^+) = \sigma(\rho^-) \equiv \sigma^{\pm}$. Finally the matching condition (5.25) implies

$$[\partial_z\tilde{\mu}_1]_{-\infty}^{+\infty} = [\nu \cdot \nabla\mu_0]_-^+ \tag{5.33}$$

and the notation $[\cdot]_-^+$ is introduced in Statement 1. Formula (5.32) together with (5.33) implies (3.12). This completes the verification of Statement 1. $\square$

Verification of Statement 2: The case $q = 3$.

In this section $\tau = \tau_3$. By assumption $\rho_0(r, 0) = \rho^{\pm}$ in $\Omega_0^{\pm}$ and the the highest order in the outer expansion of (3.2) implies that $\rho_0(r,\tau) = \rho^{\pm}$ in $\Omega_\tau^{\pm}$ also for $\tau > 0$. The lowest order in the inner expansion is still $O(\epsilon^{-2})$ and it yields once again (5.27), so that $\tilde{\rho}_0 = U$, and (5.29). The next order in the outer expansion ($O(\epsilon)$) gives

$$\partial_\tau\rho_0 = \nabla\left(\sigma_0\nabla\mu_1 + \sigma_1\nabla\mu_0\right) \tag{5.34}$$

but since $\partial_\tau\rho_0 = 0$, $\nabla\mu_0 = 0$ and $\sigma_0(\rho^{\pm}) = \sigma^{\pm} \neq 0$ we obtain

$$\Delta\mu_1 = 0 \tag{5.35}$$

Since $\partial_z\tilde{\mu}_0 = 0$ (5.29), the $O(\epsilon^{-1})$ in the inner expansion gives

$$\partial_z\left\{\tilde{\sigma}_0\partial_z\tilde{\mu}_1\right\} = 0 \tag{5.36}$$

so that $\partial_z\tilde{\mu}_1 = 0$, since $\tilde{\sigma}_0 \neq 0$ and by (5.25) $\tilde{\mu}_1(z)$ approaches (as z approaches $\pm\infty$) the value of $\nu \cdot \nabla\mu_0$ near the interface and $\nabla\mu_0 \equiv 0$. Hence

$$\tilde{\mu}_1(z, r, \tau) = C(r, \tau). \tag{5.37}$$

We will verify below that

$$\tilde{\mu}_1(z,r,\tau) = \tilde{\rho}_1(z,r,\tau)f''(\tilde{\rho}_0(z,r,\tau)) - \tilde{J} * \tilde{\rho}_1 - K(x(r))\int \tilde{J}(z-z')\tilde{\rho}_0(z',r,\tau)\mathrm{d}z' \tag{5.38}$$

and putting together (5.38) and (5.37) we obtain

$$\tilde{\rho}_1(z,r,\tau)f''(U(z)) - (\tilde{J} \star \tilde{\rho}_1)(z,r,\tau) - K(x(r))\int_{\mathbb{R}}(z-z')\tilde{J}(z-z')\mathrm{d}z' = C(r,\tau) \tag{5.39}$$

Multiply both sides of (5.39) by $U'(z)$ and integrate in the variable z over all $\mathbb{R}$. Since $\int \tilde{\rho}_1[\partial_z f'(U)/ - \tilde{J} \star U']\mathrm{d}z = 0$, the expression between square brackets being zero because it is simply $\partial_z \tilde{\mu}_0$, we obtain

$$-K(r) \int_{\mathbb{R}\times\mathbb{R}} (z - z')J(z - z')U(z')U'(z)\mathrm{d}z\mathrm{d}z' = C(r, \tau)[U]_{-\infty}^{+\infty} \tag{5.40}$$

In the left–hand side of (5.40) we recognize the surface tension of the model (3.13) and so

$$\tilde{\mu}_1 = C = -\frac{KS}{[U]_{-\infty}^{+\infty}} \tag{5.41}$$

and in particular $\tilde{\mu}_1$ is independent of time. The velocity of the interface is once again retrieved by looking at the next order in the inner expansion ($O(1)$)

$$VU' = \partial_z \left(\tilde{\sigma}_0 \partial_z \tilde{\mu}_2\right) \tag{5.42}$$

so that by integration

$$V = \frac{\sigma^{\pm} \left[\partial_z \tilde{\mu}_2\right]_{-\infty}^{+\infty}}{[U]_{-\infty}^{+\infty}} \tag{5.43}$$

and by (5.26) we finally obtain

$$V_3 \equiv V = \frac{\sigma^{\pm} \left[\nu \cdot \nabla \mu_1\right]_{-}^{+}}{[U]_{-\infty}^{+\infty}} \tag{5.44}$$

that is (3.14). On the other hand (5.25) and (5.41) give us the boundary conditions for μ_1, which then solves $\Delta\mu_1 = 0$ with

$$\mu_1(r) = -\frac{K(r)S}{[U]_{-\infty}^{+\infty}} \tag{5.45}$$

when $r \in \Gamma_\tau$, and the boundary value problem (3.15) is derived. We are left with showing that the first order correction to the chemical potential in the bulk is given by (5.38). The expression in the right–hand side of (5.38) consists of three terms: the first one is simply $f''(\tilde{\rho}_0)\tilde{\rho}_1$. The other two originate from the $O(\epsilon)$ contributions to the integral term

$$\int J_\epsilon(r - r') \left[\tilde{\rho}_0 \left(\phi(r', \Gamma_\tau)/\epsilon, r', \tau\right) + \epsilon\tilde{\rho}_1 \left(\phi(r', \Gamma_\tau)/\epsilon, r', \tau\right) + O(\epsilon^2)\right] \tag{5.46}$$

The $O(1)$ contribution to (5.46) is simply $\tilde{J} \star \tilde{\rho}_0$ and if we subtract it from (5.46) and we use $\tilde{\rho}_0 = U$ we obtain

$$\left[\int J_\epsilon(r - r')U(z')\mathrm{d}r' - \left(\tilde{J} \star U\right)(z)\right] + \epsilon\tilde{J} \star \tilde{\rho}_1 + O(\epsilon^2) \tag{5.47}$$

in which z (respectively z') is the rescaled distance of r (respectively r') from the interface, as before, i.e. $z = \phi(r, \Gamma_\tau)/\epsilon$ and $z' = \phi(r', \Gamma_\tau)/\epsilon$. We are then left to show that the term between square brackets is of order ϵ and that its value

divided by ϵ coincides to leading order with the third term in the right–hand side of (5.38). By choosing an appropriate frame of reference, we can assume that $r_i = 0$ for $i = 1, \dots, d-1$ and $r_d \equiv z\epsilon$ and that in this coordinate system the upper half plane $\{r \in \mathbb{R}^d : r_d = 0\}$ is tangent to Γ_τ. We will also orient the i–axis, $i = 1, \dots, d-1$ along the directions of principal curvatures. The curvature in the direction i will be denoted by k_i, so that $K = \sum_{i=1}^{d-1} k_i$. It is moreover convenient to go back to the original coordinates, that is to $x = r\epsilon^{-1} = (0, \dots, 0, z)$ and $x' = r'\epsilon^{-1}$: $z' = \phi(r', \Gamma_\tau)/\epsilon$ is left unchanged, since $\phi(x', \epsilon^{-1}\Gamma_\tau) = \phi(\epsilon x', \Gamma_\tau)/\epsilon$. We have

$$\int J_\epsilon(r - r')U(z')\mathrm{d}r' = \int J(x - x')U(z')\mathrm{d}x' \tag{5.48}$$

Notice that since J is supported in $B_1(0)$,the d–dimensional ball of radius 1, centered at 0, we are only interested in $|x'| = O(1)$ and $z' = O(1)$. We will parametrize $\epsilon^{-1}\Gamma_\tau$ as a function $f_\epsilon : \mathbb{R}^d \to \mathbb{R}$ around the point $0 \in \mathbb{R}^{d-1}$, that is $\{r \in \mathbb{R}^d : |(r_1, \dots, r_{d-1})|\} \cap \epsilon^{-1}\Gamma_\tau = \{(q, f_\epsilon(q)) : q \in \mathbb{R}^{d-1},\ |q| \le 1\}$. By the fact that Γ_τ is assumed to be smooth we have

$$f_\epsilon(q) = -\sum_{i=1}^{d-1} \frac{\epsilon k_i q_i^2}{2} + O(\epsilon^2) \tag{5.49}$$

for $|q| \le 1$. Our aim is to write z' explicitely as a function of x' so that it will make the integral in (5.48) more explicit. Since $x' \in \epsilon^{-1}\mathcal{N}_\epsilon$ we have

$$x' = (q, f_\epsilon(q)) + \frac{(-\nabla f_\epsilon, 1)}{\sqrt{1 + |\nabla f_\epsilon|^2}} z' + O(\epsilon^2) \tag{5.50}$$

and, by using (5.49), for $i = 1, \dots, d-1$ we obtain

$$x'_i = q_i + \epsilon k_i q_i z' + O(\epsilon^2) \tag{5.51}$$

and

$$x'_d = z' - \epsilon \sum_{i=1}^{d-1} \frac{k_i q_i^2}{2} + O(\epsilon^2) \tag{5.52}$$

which imply

$$z' = x'_d + \epsilon \sum_{i=1}^{d-1} \frac{k_i {x'_i}^2}{2} + O(\epsilon^2) \tag{5.53}$$

Let us now go back to (5.47). The term in the square brackets is equal to

$$\int J(x - x')\left[U(z') - U(x'_d)\right]\mathrm{d}x' = \epsilon \int J(x - x')\left[U'(x'_d)\sum_{i=1}^{d-1} \frac{k_i (x'_i)^2}{2}\right]\mathrm{d}x' + O(\epsilon^2) \tag{5.54}$$

in which we used the definition of $\tilde{J}$ (3.5) and (5.53). We are then left with evaluating

$$\int J(x - x')\left(\sum_i \frac{k_i (x'_i)^2}{2}\right) U'(x'_d)\mathrm{d}x' \tag{5.55}$$

in which $x = (0,\dots,0,z)$. Set $q = (x'_1,\dots,x'_{d-1}) \in \mathbb{R}^{d-1}$, $x'_d = z'$ and $R = \sqrt{q^2+(z-z')^2}$. Then the expression in (5.55) is equal to

$$
\begin{aligned}
&\frac{1}{2}\sum_i k_i \int J(R)U'(z')q_i^2 \mathrm{d}q\mathrm{d}z' \\
&= \frac{1}{2}\left(\sum_{i=1}^{d-1} k_i\right)\int J(R)U'(z')q_1^2 \mathrm{d}q\mathrm{d}z' \\
&= -\frac{K}{2}\int J'(R)\left(\frac{z'-z}{R}\right)U(z')q_1^2 \mathrm{d}q\mathrm{d}z' \qquad (5.56)\\
&= -\frac{K}{2}\int \partial_{q_1} J(R)(z'-z)U(z')q_1 \mathrm{d}q\mathrm{d}z' \\
&= \frac{K}{2}\int J(R)(z'-z)U(z')\mathrm{d}q\mathrm{d}z' \\
&= \frac{K}{2}\int (z'-z)\tilde{J}(z'-z)U(z')\mathrm{d}z'
\end{aligned}
$$

in which the first equality follows because J is rotation invariant (isotropy), the second is integration by parts with respect to z' and the fact that $K = \sum_i k_i$, the third is the chain rule of differentiation, the fourth is integration by parts with respect to q_1 and the last one follows from the definition (3.5).

This ends the verification of Statement 2. □

Appendix A: the surface tension.

This Appendix generalizes [4], which deals with the case of f given in (2.14). We are in the same framework as Proposition 4.2. The surface tension can be defined as the difference between the free energy of an equilibrium state with an interface and a homogeneous one [4,33], i.e. from (2.1)

$$
S = \lim_{L\to\infty}\frac{1}{(2L)^{d-1}}\lim_{M\to\infty}\int_{-L}^{+L}\mathrm{d}x_1\dots\int_{-L}^{+L}\mathrm{d}x_{d-1}\int_{-M}^{+M}\mathrm{d}x_d\left[g(\rho^*)(x) - f_c(\rho^+)\right] \tag{6.1}
$$

in which $\rho^*(x) = U(e_d \cdot x)$, where $e_d = (0,\dots,0,1) \in \mathbb{R}^d$, and for any measurable function $\rho : \mathbb{R}^d \to [0,1]$

$$
g(\rho)(x) = f(\rho(x)) - \frac{1}{2}[\rho(x) - 1/2](J * [\rho - 1/2])(x) \tag{6.2}
$$

Observe that $f_c(\rho^+) = f_c(\rho^-) = g(\rho^+) = g(\rho^-)$. (6.1) is well defined because of Proposition 4.2 and the properties of f. Clearly (6.1) reduces to

$$
S = \int_{\mathbb{R}}\left[g(\rho^*)(0,\dots,0,z) - f_c(\rho^+)\right]\mathrm{d}z. \tag{6.3}
$$

We are going to show that S, as defined in (6.1), coincides with (3.13).

Observe that for $\rho \in C^1$

$$\frac{\partial g}{\partial x_d}(\rho) = f'(\rho)\frac{\partial \rho}{\partial x_d} - \frac{1}{2}\frac{\partial \rho}{\partial x_d} J * [\rho - 1/2] - \frac{1}{2}[\rho - 1/2] J * \frac{\partial \rho}{\partial x_d} \tag{6.4}$$

but, by (3.6), $f'(\rho^*) = J * (\rho^* - 1/2)$. Hence

$$\begin{aligned} \frac{\partial g}{\partial x_d}(\rho^*) &= \frac{1}{2}\frac{\partial \rho^*}{\partial x_d} J * [\rho^* - 1/2] - \frac{1}{2}[\rho^* - 1/2] J * \frac{\partial \rho^*}{\partial x_d} \\ &= \frac{1}{2}\Big(U'(x_d)(\tilde{J} \star [U - 1/2])(x_d) - [U(x_d) - 1/2](\tilde{J} \star U')(x_d)\Big) \end{aligned} \tag{6.5}$$

Integrating (6.3) by parts and substituting (6.5) in the integrand we finally obtain

$$\begin{aligned} S &= -\frac{1}{2}\int_{\mathbb{R}} x_d \frac{\partial g}{\partial x_d}(\rho^*) \mathrm{d}x_d \\ &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} (z' - z) U'(z) \tilde{J}(z - z')[U(z') - 1/2] \mathrm{d}z \mathrm{d}z' \\ &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} (z' - z) U'(z) \tilde{J}(z - z') U(z') \mathrm{d}z \mathrm{d}z' \end{aligned} \tag{6.6}$$

where we used the fact that there is an $m' > 0$ such that $\lim_{z \to \pm\infty} e^{m'|z|} U'(z) = 0$, which follows from (3.6) and Proposition 4.2, point (3). Since (6.6) is (3.13), we are done.

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